These are some sample problems taken from exams in previous years. Your exam will have roughly ten questions.

1. (30 points) Circle T or F
   T F Any curve that obeys the variation diminishing property also obeys the convex hull property.
   T F If two cubic Bézier curves that meet with $C^2$ continuity are degree elevated, they will meet with $C^3$ continuity.
   T F The process of inserting a knot into a B-spline curve is sometimes called Horner’s algorithm.
   T F A single Bézier curve can be represented as a special case of a B-spline curve.
   T F Any curve that obeys the convex hull property is also variation diminishing.
   T F The polar form of $f(t) = 1 + 2t + 3t^2$ is $f(t_1, t_2) = 1 + 2(t_1 + t_2) + 3t_1t_2$.
   T F A periodic degree-3 B-Spline curve with $n$ control points and for which all knot intervals are non-zero is comprised of $n - 3$ Bézier curves.
   T F A degree-4 B-spline curve $P(t)$ has a knot vector $[0, 0, 0, 0, 1, 3, 4, 4, 4]$. The Cartesian coordinates of $P(2)$ can be found by performing three knot insertions.
   T F Take a cubic polynomial Bézier curve $P(t)$ and assign weights to its control points of 1, 2, 4, and 8, respectively. Call the resulting rational Bézier curve $Q(t)$. Then $P(t) \equiv Q(t)$ for all values of $t$, $0 \leq t \leq 1$.
   T F It is possible to draw an entire explicit B-Spline curve using a single forward difference table as long as the knots are evenly spaced.
   T F Overhauser curves obey the convex hull property.
   T F The three-dimensional vectors $(1, 0, 2)$ and $(2, -1, 0)$ are perpendicular.
   T F The offset of a any degree-three Bézier curve can be exactly represented as a degree-eight rational Bézier curve.
   T F If you add a constant to all of the knot intervals in a periodic B-Spline curve, the appearance of the curve will not change.
   T F If you scale all of the knot intervals in a periodic B-Spline curve by a constant, the appearance of the curve will not change.
2. The Bizet curve is given by \( \mathbf{P}(t) = (1-t)^3\mathbf{P}_0 + 2t(1-t)\mathbf{P}_1 + t(1-t)\mathbf{P}_2 + t^3\mathbf{P}_3. \)

a. **YES** NO Does this curve interpolate the endpoints? Why or why not?

b. **YES** NO Is this curve symmetric? Why or why not?

c. What is the derivative vector at \( t=0 \) and \( t=1 \)?
   \[ P'(0) = \quad P'(1) = \]

d. **YES** NO Is this curve coordinate system independent? Why or why not?

e. **YES** NO Does this curve obey the convex hull property?

3. Convert the power basis rational curve
   \[ x = \frac{2t^2}{t^2 + 1} \quad y = \frac{(t + 1)^2}{t^2 + 1} \]
to rational Bézier form. That is, find the control points and weights for the equivalent rational Bézier curve.
   Answer: \( \mathbf{P}_0 = ( \quad , \quad )w_0 = \quad ; \quad \mathbf{P}_1 = ( \quad , \quad )w_1 = \quad ; \quad \mathbf{P}_2 = ( \quad , \quad )w_2 = \quad . \)

4. Find the coordinates of the point \((17, 34, 21)\) after rotating it \( \theta = 36.8699^\circ \) about the axis that goes through point \((6, 21, 17)\) with direction vector \((2, 1, -2)\).
   Take \( \sin(\theta) = \frac{3}{5} \) and \( \cos(\theta) = \frac{4}{5} \).

5. Find the control points of the cubic Bézier curve for which
   \[ \mathbf{P}(0) = (0, 0); \quad \mathbf{P'} = (6, 6); \quad \mathbf{P''} = (0, -12); \quad \mathbf{P'''} = (-6, 6). \]

6. Find the curvature at \( t = \frac{1}{2} \) for the degree two rational Bézier curve \( \mathbf{P}_{[0,1]}(t) \) with control points and weights
   \[ \mathbf{P}_0 = (10, 0), \quad w_0 = 1; \quad \mathbf{P}_1 = (0, 0), \quad w_1 = 1; \quad \mathbf{P}_2 = (0, 10), \quad w_2 = 2. \]
7. Find the control points of the B-spline with knot vector 

\[ [0001222] \]

given the following polar values

\[ f(0, 0, 0) = (0, 0); \quad f(0, 0, 1) = (0, 4); \quad f(0, 1, 1) = (4, 8); \]
\[ f(1, 1, 1) = (8, 8); \quad f(2, 2, 2) = (12, 4). \]

8. A degree-three curve equation is given using symmetric polynomials as

\[ P(t_1, t_2, t_3) = (1, 2) + (3, 4) \frac{t_1 + t_2 + t_3}{3} + (5, 6) \frac{t_1t_2 + t_1t_3 + t_2t_3}{3} + (7, 8)t_1t_2t_3 \]

a. Give a power-basis equation for the curve \( P(t) = P(t, t). \)

b. If we wanted to represent that curve as a B-spline curve with a knot vector [0, 0, 0, 3, 6, 9], what would the Cartesian coordinates of the control points be?

9. What is the parameter value \( t \in [0, 1] \) at which the rational cubic Bézier curve \( P_{[0,1]}(t) \) with control points and weights

\[ P_0 = (0, 0, 4); \quad P_1 = (1, 1, 2); \quad P_2 = (5, 5, 2); \quad P_3 = (7, 7, 3) \]

intersects the line \( x + y - 6 = 0. \)

10. A certain degree two B-spline curve has a knot vector [0, 0, 1, 1, 3, 5, 7, 9, 9].
   a. For the control point whose polar label is \( f(3, 5) \), sketch its blending function as an explicit B-spline curve. Show the Cartesian coordinates of the control points.
   b. Evaluate this blending function at \( t = 4. \)
   c. For what values of \( t \) is this blending function non-zero?
11. Give the equivalent power-basis equations $x(t)$, $y(t)$ for the cubic Bézier curve with control points:

$$P_0 = (1,3), \quad P_1 = (3,3), \quad P_2 = (5,8), \quad P_3 = (7,6).$$

12. A degree four B-spline has a knot vector $[a \ b \ c \ d \ e \ f \ g \ h \ i \ j \ k \ l \ m \ n \ o \ p \ q \ r \ s]$. None of the knots are multiple. If the control point $(c \ d \ e \ f)$ is moved, what are the parameter ranges of the underlying Bézier curves that are changed?

13. Convert to degree four Bernstein basis the polynomial

$$t^3 + 1.$$

Answer:

$$B_0^4(t) + B_1^4(t) + B_2^4(t) + B_3^4(t).$$

14. a. If the weights $w_1$ and $w_2$ can be any non-negative value, give an upper and lower bound for the area of the shaded region. State which property of Bézier curves you are using to arrive at your bounds.

b. If $w_1 = w_2 = 1$, what is the area of the shaded region?

15. Recall that the equation for Ball’s cubic curve is:

$$(1 - t)^2 Q_0 + 2t(1 - t)^2 Q_1 + 2t^2(1 - t) Q_2 + t^2 Q_3.$$

Given a Ball curve with control points $Q_0$, $Q_1$, $Q_2$, $Q_3$, find the control points $P_0$, $P_1$, $P_2$, $P_3$ of an equivalent cubic Bézier curve. (Hint: write down what you know about the relationship between Ball curves and Bézier curves.)
16. This degree four polynomial Bézier curve, which begins at \( t = 1 \) and ends at \( t = 4 \), is also a B-spline with knot vector \([11114444]\). Insert a knot at \( t = 2 \).

\[ P_0 = (1,1) \quad P_1 = (1,4) \quad P_2 = (4,7) \quad P_3 = (7,4) \quad P_4 = (7,1) \]

State the control point coordinates and knot vector after this knot insertion is performed.

17. For the degree 3 polynomial \( f(t) \), we have:

\[ f(1) = 3; \quad f(2) = 6; \quad f(4) = 4; \quad f(5) = 17. \]

Find \( f(3) \) using forward differencing.

18. The weight ratio of a rational Bézier curve is the largest divided by the smallest weight. For example, the weight ratio of a cubic Bézier curve with weights 8, 2, 3, 4 is 4. If you want to plot a rational Bézier curve using a fixed number of evenly-spaced line segments, the resulting plot will generally look smoother if the curve is reparameterized so that weight ratio is minimized. For example, a rational cubic Bézier curve with weights 8, 2, 3, 1 will not generally look as smooth as one with weights 2, 1, 3, 2.

Given a rational cubic Bézier curve with weights 1, 3, 7, 15, reparameterize the curve so that the weight ratio is less than 2.

19. A degree one NURBS curve has knot vector \([1, 3, 5, 7]\) and control points and weights:

\[ P_0 = (1,1); w_0 = 1. \quad P_1 = (3,3); w_1 = 1. \quad P_2 = (9,7); w_2 = 3. \quad P_3 = (10,2); w_3 = 4. \]

Insert a knot at \( t = 4 \). Your answer should state the new knot vector and the new control points.

20. Find the control points \( P_1 \) and \( P_2 \) of a polynomial cubic Bézier curve which is curvature continuous with the circles of radius 2 and 4 as shown.
21. Find all possible coordinates for control points $Q_1$, $Q_2$, and $Q_3$ such that these two Bézier curves are $C^2$. (Note that $P(t)$ is a degree two polynomial Bézier curve and $Q(t)$ is a polynomial cubic Bézier curve).