

## Chapter 3

# Polynomial Evaluation and Basis Conversion

### 3.1 Horner's Algorithm in Power Basis

Horner's algorithm can evaluate a power basis polynomial

$$y(t) = \sum_{i=0}^n p_i t^i$$

using  $n$  multiplies and adds. The trick is to rewrite the polynomial in nested form:

$$y(t) = (\cdots((p_n t + p_{n-1})t + p_{n-2})t + \cdots + p_1)t + p_0$$

This can be written

$$\begin{aligned} h_n &= p_n \\ h_{i-1} &= t \cdot h_i + p_{i-1}, \quad i = n, \dots, 1 \\ y(t) &= h_0 \end{aligned}$$

or, as a C function,

```
float p_horner(float *p,int n, float t)
{
float h;
int i;

    h = p[n];
    for(i=n-1; i>=0; i--)
        h = t*h + p[i];
    return h;
}
```

This is the fastest algorithm for evaluating a power basis polynomial at a single point.

For evaluating a polynomial at several evenly spaced intervals ( $f(0)$ ,  $f(a)$ ,  $f(2a)$ ,  $f(3a)$ , etc.) forward differencing works faster (see Chapter 4) but suffers from numerical instability.

## 3.2 Horner's Algorithm in Bernstein Basis

We have seen that one way to evaluate a Bézier curve (that is, to compute the Cartesian coordinates of a point on a Bézier curve) is to use the de Casteljau algorithm. Another option is to convert it to power basis (see Section 3.3) and then use Horner's algorithm in Section 3.1. Alternatively, Horner's algorithm can be modified to work directly on Bernstein polynomials. We can use the recurrence relation for binomial coefficients

$$\binom{n}{i} = \frac{n-i+1}{i} \binom{n}{i-1}$$

to derive a nested multiplication which does not involve the binomial coefficients directly:

```
float b_horner(float *b, int n, float t)
{
float u, bc, tn, tmp;
int i;

    u = 1.0 - t;
    bc = 1;
    tn = 1;
    tmp = b[0]*u;
    for(i=1; i<n; i++){
        tn = tn*t;
        bc = bc*(n-i+1)/i;
        tmp = (tmp + tn*bc*b[i])*u;
    }
    return (tmp + tn*t*b[n]);
}
```

## 3.3 Basis Conversion

A fundamental problem in CAGD is that of polynomial basis conversion. We here consider the common case of conversion between power and Bernstein bases. For example, given a Bézier curve of any degree, we want to find the equivalent power basis equation and vice versa.

Denote a polynomial in Bernstein form

$$B(t) = \sum_{i=0}^n b_i \binom{n}{i} (1-t)^{n-i} t^i$$

and a polynomial in power form

$$P(t) = \sum_{i=0}^n p_i t^i.$$

The problem we consider is, given the  $b_i$  of a polynomial in Bernstein basis, find the  $p_i$  for the equivalent power basis polynomial (Bernstein to power basis conversion) or given the  $p_i$  find the  $b_i$  (power to Bernstein conversion).

An elegant solution to this problem can be obtained by performing a Taylor's series expansion. The Taylor's expansion of  $B(t)$  is:

$$B(t) \equiv B(0) + B'(0)t + \frac{B''(0)t^2}{2} + \dots + \frac{B^{(i)}(0)t^i}{i} + \dots$$

A power basis polynomial is equivalent to a Bernstein basis polynomial ( $P(t) \equiv B(t)$ ) if and only if

$$\frac{P^{(i)}(0)t^i}{i!} \equiv \frac{B^{(i)}(0)t^i}{i!}, \quad i = 0, \dots, n.$$

But, for power basis,

$$\frac{P^{(i)}(0)}{i!} = p_i$$

so

$$p_i = \frac{B^{(i)}(0)}{i!}, \quad i = 0, \dots, n. \quad (3.1)$$

The terms  $B^{(i)}(0)$  can be found by setting up a difference table. Remember from our study of hodographs that the coefficients of the derivative of a polynomial in Bernstein form (for example, the  $x$  or  $y$  component of a Bézier curve) are:

$$n(b_1 - b_0), \quad n(b_2 - b_1), \dots, n(b_n - b_{n-1}).$$

The coefficients of the second derivative are:

$$n(n-1)(b_2 - 2b_1 + b_0), \quad n(n-1)(b_3 - 2b_2 + b_1), \dots, n(n-1)(b_n - 2b_{n-1} + b_{n-2}).$$

Since  $B(0) = \sum_{i=0}^n b_i \binom{n}{i} (1-0)^{n-i} 0^i = b_0$ , we have

$$B'(0) = n(b_1 - b_0), \quad B''(0) = n(n-1)(b_2 - 2b_1 + b_0)$$

$$B^{(i)}(0) = n(n-1) \cdots (n-i+1) \sum_{j=0}^i (-1)^{(i-j+1)} \binom{i}{j} b_j.$$

This can be written more neatly if we define the recurrence

$$b_i^j = b_{i+1}^{j-1} - b_i^{j-1}$$

with  $b_i^0 \equiv b_i$ . Then

$$B^{(i)}(0) = n(n-1) \cdots (n-i+1) b_0^i = \frac{n!}{(n-i)!} b_0^i.$$

From equation 21.1,

$$p_i = \frac{n!}{(n-i)! i!} b_0^i = \binom{n}{i} b_0^i.$$

Thus, the problem reduces to one of finding the values  $b_0^i$ . This is easily done using a difference table:

|  |                                     |         |                             |
|--|-------------------------------------|---------|-----------------------------|
| $b_0^0 = b_0 = p_0$  | $b_1^0 = b_1$                       | $\dots$ | $b_n^0 = b_n$               |
| $b_0^1 = b_1^0 - b_0^0 = p_1 / \binom{n}{1}$                   | $b_1^1 = b_2^0 - b_1^0$             | $\dots$ | $b_n^1 = b_n^0 - b_{n-1}^0$ |
| $b_0^2 = b_1^1 - b_0^1 = p_2 / \binom{n}{2}$                   | $b_1^2 = b_2^1 - b_1^1$             | $\dots$ | $b_n^2 = b_n^1 - b_{n-1}^1$ |
| $\dots$  | $\dots$                             | $\dots$ | $\dots$                     |
| $b_0^{n-1} = b_1^{n-2} - b_0^{n-2} = p_{n-1} / \binom{n}{n-1}$ | $b_1^{n-1} = b_2^{n-2} - b_1^{n-2}$ | $\dots$ | $\dots$                     |
| $b_0^n = b_1^{n-1} - b_0^{n-1} = p_n$                          |                                     |         |                             |

Thus, to perform Bernstein to power basis conversion, load the Bernstein coefficients into the top row and compute the difference table. Scale the left column by  $\binom{n}{i}$ , and you have the power coefficients.

To perform power to Bernstein conversion, divide the power coefficients by  $\binom{n}{i}$ , load them into the left column, compute the difference table backwards, and read the Bernstein coefficients off the top row.

### 3.3.1 Example: Convert Bernstein-Basis Polynomial to Power Basis

Convert to power basis the degree 4 Bernstein form polynomial with coefficients (1, 3, 4, 6, 8). This is done by setting up the difference table

$$\begin{array}{cccccc} 1 & 3 & 4 & 6 & 8 & \\ 2 & 1 & 2 & 2 & & \\ -1 & 1 & 0 & & & \\ 2 & -1 & & & & \\ -3 & & & & & \end{array}$$

so the power coefficient are taken from the left column, times the binomial coefficients:

$$p_0 = 1 \binom{4}{0} = 1, \quad p_1 = 2 \binom{4}{1} = 8, \quad p_2 = -1 \binom{4}{2} = -6, \quad p_3 = 2 \binom{4}{3} = 8, \quad p_4 = -3 \binom{4}{4} = -3$$

### 3.3.2 Example: Convert Power Basis Rational Equations to Bézier Form

Express the rational curve

$$x = \frac{x(t)}{w(t)} = \frac{20t^3 - 39t^2 + 24t + 1}{7t^3 - 12t^2 + 6t + 1}, \quad y = \frac{y(t)}{w(t)} = \frac{27t^3 - 57t^2 + 30t + 2}{7t^3 - 12t^2 + 6t + 1},$$

as a rational Bézier curve. Begin by placing the power-basis coefficients in the left column, divide by  $\binom{n}{i}$ , then complete the difference table:

$$\begin{array}{l} (1, 2, 1) / \binom{3}{0} = \\ (24, 30, 6) / \binom{3}{1} = \\ (-39, -57, -12) / \binom{3}{2} = \\ (20, 27, 7) / \binom{3}{3} = \end{array} \left| \begin{array}{cccc} (1, 2, 1) & (9, 12, 3) & (4, 3, 1) & (6, 2, 2) \\ (8, 10, 2) & (-5, -9, -2) & (2, -1, 1) & \\ (-13, -19, -4) & (7, 8, 3) & & \\ (20, 27, 7) & & & \end{array} \right.$$

The control points and weights are found on the top row (the weights are multiplied times the control point coordinates). Thus, for this example,

$$\mathbf{P}_0 = (1, 2), \quad w_0 = 1; \quad \mathbf{P}_1 = (3, 4), \quad w_1 = 3; \quad \mathbf{P}_2 = (4, 3), \quad w_2 = 1; \quad \mathbf{P}_3 = (3, 1), \quad w_3 = 2.$$

### 3.3.3 Closed Form Expression

The conversion from Bernstein to power basis can be written concisely as follows:

$$p_i = \sum_{k=0}^i b_k \binom{n}{i} \binom{i}{k} (-1)^{i-k}.$$